$$\delta h_1^2 = 0.2953, \quad \delta h_2^2 = 0.2095, \quad \delta h_3^2 = 0.2096;$$

compression behind wave $\rho_1/\rho_0 = 1.05$

$$\delta h_1^2 = 0.3184, \quad \delta h_2^2 = 0.2216, \quad \delta h_3^2 = 0.2217.$$

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LITERATURE CITED

1. S. K. Godunov and E. I. Romenskii, "The non-steady-state equations of the nonlinear equations of the theory of elasticity in Euler coordinates," Zh. Prikl. Mekh. Tekh. Fiz., No. 5 (1972).
2. S. K. Godunov, V. V. Denisenko, N. S. Kozin, and N. K. Kuz'mina, "Use of a relaxation model of viscosity with calculation of monaxial homogeneous deformations and refinement of the interpolation formulas of Maxwell viscosity," Zh. Prikl. Mekh. Tekh. Fiz., No. 5 (1975).
3. S. K. Godunov, N. S. Kozin, and E. I. Romenskii, "Equation of state of the elastic energy of metals with a nonspherical tensor of the deformations," Zh. Prikl. Mekh. Tekh. Fiz., No. 2 (1974).
4. S. K. Godunov and N. S. Kozin, "The structure of shock waves with a nonlinear dependence of the Maxwell viscosity on the parameters of the substance," Zh. Prikl. Mekh. Tekh. Fiz., No. 5 (1974).
5. A. A. Deribas, V. S. Zakharov, T. M. Sobolenko, and T. S. Teslenko, "The transfer of the surface relief in metals by shock waves," Fiz. Goreniya Vzryva, No. 6 (1974).

EQUATIONS OF THE LINEAR THEORY OF ELASTICITY WITH POINT MAXWELLIAN SOURCES OF STRESS RELAXATION

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1. General Solution. Relationships on the Characteristic

The system being studied has the form

$$\begin{gathered}
\rho_0 U \frac{\partial u}{\partial x} - \frac{\partial \sigma_{11}}{\partial x} - \frac{\partial \sigma_{12}}{\partial y} = 0, \\
\rho_0 U \frac{\partial v}{\partial x} - \frac{\partial \sigma_{12}}{\partial x} - \frac{\partial \sigma_{22}}{\partial y} = 0, \\
U \frac{\partial \sigma_{11}}{\partial x} - \rho_0 c_0^2 \frac{\partial u}{\partial x} - \rho_0 (c_0^2 - 2b_0^2) \frac{\partial v}{\partial y} = 0, \\
U \frac{\partial \sigma_{22}}{\partial x} - \rho_0 (c_0^2 - 2b_0^2) \frac{\partial u}{\partial x} - \rho_0 c_0^2 \frac{\partial v}{\partial y} = 0, \\
U \frac{\partial \sigma_{33}}{\partial x} - \rho_0 (c_0^2 - 2b_0^2) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0, \\
U \frac{\partial \sigma_{12}}{\partial x} - \rho_0 b_0^2 \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = 0,
\end{gathered} \tag{1.1}$$

where σ_{11}, σ_{22}, σ_{33}, σ_{12} are the components of the stress tensor; u + U is the horizontal component of the vector of the displacement rate of points of the medium ($U < b_0 < c_0 \ll U$); v is the vertical component of the vector of the displacement rate of points of the medi-

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um; ρ_0 is the density of the medium; c_0 is the longitudinal velocity of sound; b_0 is the transverse velocity of sound.

The system (1.1) can be transformed in such a way that for the four functions $\sigma_{11}(x, y)$, $\sigma_{22}(x, y)$, $\sigma_{33}(x, y)$, $\sigma_{12}(x, y)$ we obtain the system

$$
\begin{aligned}
&\frac{\partial\sigma_{12}}{\partial y} = \rho_0 U \frac{\partial v}{\partial x} - \frac{\partial\sigma_{12}}{\partial x},\\
&\frac{\partial\sigma_{22}}{\partial y} = \frac{\rho_0\left(c_0^2U^2 - 4b_0^2c_0^2 + 4b_0^4\right)}{c_0^2U}\frac{\partial u}{\partial x} - \frac{c_0^2 - 2b_0^2}{c_0^2}\frac{\partial\sigma_{12}}{\partial x},\\
&\frac{\partial u}{\partial y} = \frac{U}{\rho_0 b_0^2}\frac{\partial\sigma_{12}}{\partial x} - \frac{\partial v}{\partial x},\\
&\frac{\partial v}{\partial y} = \frac{U}{\rho_0 c_0^2}\frac{\partial\sigma_{22}}{\partial x} - \frac{c_0^2 - 2b_0^2}{c_0^2}\frac{\partial u}{\partial x},
\end{aligned}
\tag{1.2}
$$

which has to be supplemented on the twofold characteristics y = const:

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left[\sigma_{11} - \frac{4\rho_0 b_0^2}{U}\left(\frac{c_0^2 - b_0^2}{c_0^2}\right)u - \left(\frac{c_0^2 - 2b_0^2}{c_0^2}\right)\sigma_{22}\right] = 0,\\
&\frac{\partial}{\partial x}\left[\sigma_{33} - \frac{2\rho_0 b_0^2}{U}\left(\frac{c_0^2 - 2b_0^2}{c_0^2}\right)u - \left(\frac{c_0^2 - 2b_0^2}{c_0^2}\right)\sigma_{22}\right] = 0.
\end{aligned}
\tag{1.3}
$$

The system (1.2) is of the elliptic type; its general solution is represented in terms of two analytic functions $F(z_1)$ and $P(z_2)$ of the arguments z_1 and z_2, where $z_1 = x + i\omega_1 y$; $z_2 = x + i\omega_2 y$;

$$\omega_1 = \sqrt{1 - \frac{U^2}{c_0^2}}; \quad \omega_2 = \sqrt{1 - \frac{U^2}{b_0^2}}.$$

The real and imaginary parts of the functions

$$F(z_1) = f(x, y) + ig(x, y), \; P(x, y) = p(x, y) + iq(x, y)$$

are connected by relations of the Cauchy–Riemann type

$$
\begin{aligned}
&\frac{\partial f}{\partial x} = \frac{1}{\omega_1}\frac{\partial g}{\partial y}, \quad \frac{\partial p}{\partial x} = \frac{1}{\omega_2}\frac{\partial q}{\partial y},\\
&\frac{\partial f}{\partial y} = -\omega_1\frac{\partial g}{\partial x}, \quad \frac{\partial p}{\partial y} = -\omega_2\frac{\partial q}{\partial x}.
\end{aligned}
$$

The solution of (1.2) has the form

$$
\begin{aligned}
\sigma_{22}(x, y) &= -\operatorname{Im}\left\{\frac{1-\omega_2^2}{2\omega_1}F(z_1) + \frac{2\omega_2}{1+\omega_2^2}P(z_2)\right\},\\
\sigma_{12}(x, y) &= \operatorname{Re}\{F(z_1) + P(z_2)\},\\
u(x, y) &= \frac{U}{2\rho_0 b_0^2}\operatorname{Im}\left\{\frac{1}{\omega_1}F(z_1) + \frac{2\omega_2}{1-\omega_2^2}P(z_2)\right\},\\
v(x, y) &= \frac{U}{2\rho_0 b_0^2}\operatorname{Re}\left\{F(z_1) + \frac{2}{1+\omega_2^2}P(z_2)\right\}.
\end{aligned}
\tag{1.4}
$$

The representation of the solution in terms of two analytic functions is analogous to the representation given in [1], with the difference that in [1] the unknowns are not the displacement rates of points of the medium but the displacements themselves.

The relations (1.3) give us the possibility of determining $\sigma_{11}(x, y)$, $\sigma_{33}(x, y)$ from $\sigma_{22}(x, y)$ and $u(x, y)$.

2. Interpretation of the Right Sides

Let a source, describable by delta-formed right sides of the equation system being studied, be concentrated at the origin of the coordinates of the plane (x, y), which moves at a subsonic velocity. It is necessary to determine the stresses and displacement rates of points of the medium.

The system of equations in this case has the form

$$\begin{gathered}
\rho_0 U \frac{\partial u}{\partial x} - \frac{\partial \sigma_{11}}{\partial x} - \frac{\partial \sigma_{12}}{\partial y} = R\delta(x)\,\delta(y), \\
\rho_0 U \frac{\partial v}{\partial x} - \frac{\partial \sigma_{12}}{\partial x} - \frac{\partial \sigma_{22}}{\partial y} = G\delta(x)\,\delta(y), \\
U \frac{\partial \sigma_{11}}{\partial x} - \rho_0 c_0^2 \frac{\partial u}{\partial x} - \rho_0 \left(c_0^2 - 2b_0^2\right) \frac{\partial v}{\partial y} = \varphi_1 \delta(x)\,\delta(y), \\
U \frac{\partial \sigma_{22}}{\partial x} - \rho_0 \left(c_0^2 - 2b_0^2\right) \frac{\partial u}{\partial x} - \rho_0 c_0^2 \frac{\partial v}{\partial y} = \varphi_2 \delta(x)\,\delta(y), \\
U \frac{\partial \sigma_{33}}{\partial x} - \rho_0 \left(c_0^2 - 2b_0^2\right) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \varphi_3 \delta(x)\,\delta(y), \\
U \frac{\partial \sigma_{12}}{\partial x} - \rho_0 b_0^2 \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \varphi_4 \delta(x)\,\delta(y).
\end{gathered} \tag{2.1}$$

The presence of the right sides in the system of equations (2.1) leads to the fact that right sides appear in the relations (1.3) and, consequently, $\sigma_{11}(x, y)$ and $\sigma_{33}(x, y)$ are determined from $u(x, y)$ and $\sigma_{22}(x, y)$ as follows:

$$\begin{gathered}
\sigma_{11}(x, y) = \operatorname{Im} \left\{ \frac{1 - \omega_2^2 + 2\omega_1^2}{2\omega_1} F(z_1) + \frac{2\omega_2}{1 + \omega_2^2} P(z_2) \right\} + N\theta(x)\,\delta(y), \\
\sigma_{33}(x, y) = \operatorname{Im} \left\{ \frac{2\omega_1^2 - \omega_2^2 - 1}{2\omega_1} F(z_1) \right\} + M\theta(x)\,\delta(y),
\end{gathered} \tag{2.2}$$

where $N = \frac{1}{U}\left(\varphi_1 - \frac{c_0^2 - 2b_0^2}{c_0^2}\varphi_2\right)$;

$$M = \frac{1}{U}\left(\varphi_3 - \frac{c_0^2 - 2b_0^2}{c_0^2}\varphi_1\right);$$

$$\theta(x) = \int_{-\infty}^{x} \delta(\xi)\,d\xi.$$

Everywhere, except in the neighborhood of the origin of the coordinates, σ_{22}, σ_{12}, u, v in the solution of the system (2.1) can be determined from the expressions (1.4), while the concentrated right sides (for $x = 0$, $y = 0$) can be satisfied, choosing $F(z_1)$, $P(z_2)$ in the form $F(z_1) = (A + iB)/z_1$, $P(z_2) = (C + iD)/z_2$,

where

$$A = \frac{\omega_1}{1 - \omega_2^2}\left[\frac{2\varphi_4}{\pi U} + \frac{G}{\pi}\right]; \quad B = \frac{1}{1 - \omega_2^2}\left[-\frac{R + N}{\pi} + \frac{\left(1 + \omega_2^2\right)\varphi_2 b_0^2}{\pi U c_0^2}\right];$$

$$C = -\frac{1 - \omega_2^2}{1 + \omega_2^2}\left[\frac{\varphi_4}{2\omega_1 \pi U} + \frac{1}{2\omega_2}\frac{G}{\pi}\right]; D = \frac{1 + \omega_2^2}{1 - \omega_2^2}\left[\frac{R + N}{2\pi} - \frac{\varphi_2}{\pi U}\frac{b_0^2}{c_0^2}\right].$$

If the behavior of the medium is described by the Maxwellian relaxation model, then the system of equations has the form

$$\rho_0 U \frac{\partial u}{\partial x} - \frac{\partial \sigma_{11}}{\partial x} - \frac{\partial \sigma_{12}}{\partial y} = R\delta(x)\delta(y),$$

$$\rho_0 U \frac{\partial v}{\partial x} - \frac{\partial \sigma_{12}}{\partial x} - \frac{\partial \sigma_{22}}{\partial y} = G\delta(x)\delta(y),$$

$$U\frac{\partial \sigma_{11}}{\partial x} - \rho_0 c_0^2 \frac{\partial u}{\partial x} - \rho_0 (c_0^2 - 2b_0^2)\frac{\partial v}{\partial y} = -\frac{1}{\tau}\left(\sigma_{11} - \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}\right),$$

$$U\frac{\partial \sigma_{22}}{\partial x} - \rho_0 (c_0^2 - 2b_0^2)\frac{\partial u}{\partial x} - \rho_0 c_0^2 \frac{\partial v}{\partial y} = -\frac{1}{\tau}\left(\sigma_{22} - \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}\right),$$

$$U\frac{\partial \sigma_{33}}{\partial x} - \rho_0 (c_0^2 - 2b_0^2)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{1}{\tau}\left(\sigma_{33} - \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}\right),$$

$$U\frac{\partial \sigma_{12}}{\partial x} - \rho_0 b_0^2 \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -\frac{\sigma_{12}}{\tau},$$

where τ is the relaxation time of shear stresses, being, as a rule, a suddenly varying function of the stress state.

We assume that the applied forces or some other causes give rise to plastic strains that are describable by the Maxwellian model only in a small neighborhood of the origin of the coordinates $x = 0$, $y = 0$. This assumption can be modeled, putting $1/\tau$ equal to zero outside the circle $\{x^2 + y^2 \leqslant \varepsilon^2\} = K$ and equal within the circle to the effective value $1/\tau_s$, where τ_s is the characteristic relaxation time in the plastic zone.

We shall assume that φ_1, φ_2, φ_3, φ_4 constitute "powers" of relaxation that are "summary" over the entire plasticity zone; their values will be modeled by the following integrals:

$$\varphi_i = -\iint\limits_K \left(\sigma_{ii} - \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}\right)\frac{dxdy}{\tau_s} \qquad (i = 1, 2, 3),$$
$$\varphi_4 = -\iint\limits_K \frac{\sigma_{12}}{\tau_s}\, dxdy. \tag{2.3}$$

When using the scheme developed here for the estimates of the numerical perturbation values, which are introduced by the presence of the plasticity zone, we can calculate the integrals (2.3) from the solution of the elastic problem, i.e., from the solution of the problem with $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = 0$.

We see that in the model under consideration the relation

$$\varphi_1 + \psi_2 + \varphi_3 = 0 \tag{2.4}$$

is fulfilled.

Thus, if the behavior of the medium is described by means of the Maxwellian model with point sources of relaxation, then the right sides of the system (2.1) have the following meaning: φ_1, φ_2, φ_3 are the powers of the relaxation sources of the normal stresses; φ_4 is the power of the relaxation source of the shear stress; (R, G) is the power of the vector of external force applied to the medium in the neighborhood of the origin of the coordinates.

The stresses σ_{11} and σ_{33} are given by the expressions (2.2), from which we see that for nonzero powers of the relaxation sources of normal stresses, the stress trace remains behind the singular point along the real axis. This trace is characterized by the quantities N and M.

3. An Estimate of the Stressed Layer Occurring under the Action of a Point Die

We consider the problem of determining the field of stresses and velocities in a half-plane ($y < 0$) moving at a subsonic velocity U, if we know that the boundary of the half-plane is free from stresses $\sigma_{12}(x, 0) = \sigma_{22}(x, 0) = 0$, while in the neighborhood of the origin of the coordinates a force with components (R, G) is applied to it, and there exists a point source of stress relaxation modeling plastic strains close to the application point of the force.

The stress state of the half-plane is described by the system (2.1). The boundary conditions and the right sides concentrated in the neighborhood of the origin of the coordinates are satisfied if in the expressions (1.4), (2.2) we put $F(z_1)$ and $P(z_2)$ equal to

$$F(z_1) = \frac{1}{\pi A_0 z_1}\left[\frac{1+\omega_2^2}{2\omega_1} G - 2i(R+N)\right],$$
$$P(z_2) = \frac{1}{\pi A_0 z_2}\left[-\frac{1+\omega_2^2}{2\omega_2} G + 2i(R+N)\right],$$

where

$$A_0 = 1 - \frac{(1+\omega_2^2)^2}{4\omega_1\omega_2}.$$

Behind the application point of the force along the boundary of the half-plane (on the positive part of the real axis) there arises a stress trace. The appearance of this stress trace is caused by the presence of the point source of relaxation of normal stresses in the neighborhood of the application point of the force. From the solution thus obtained it is seen that the stress trace has the same effect on the elastic solution, i.e., on the solution in which we do not take into account stress relaxation, as a force applied in the neighborhood of the origin of the coordinates having the components (N, 0).

Let the square of the rate of motion of the half-plane be considerably less than the square of the velocity of the transverse sound waves, i.e., $U^2 \ll b_0^2$. In this case we can obtain an expansion of the solution by powers of the ratio U^2/b_0^2. The terms with zero degree of the expansion parameter have the form

$$\begin{gathered}
\sigma_{11}^0(x, y) = -\frac{2G}{\pi}\frac{x^2 y}{(x^2+y^2)^2} - \frac{2(R+N)}{\pi}\frac{x^3}{(x^2+y^2)^2} + N\theta(x)\delta(y),\\
\sigma_{22}^0(x, y) = -\frac{2G}{\pi}\frac{y^3}{(x^2+y^2)^2} - \frac{2(R+N)}{\pi}\frac{xy^2}{(x^2+y^2)^2},\\
\sigma_{33}^0(x, y) = -\left(\frac{c_0^2-2b_0^2}{c_0^2-b_0^2}\right)\frac{G}{\pi}\frac{y}{(x^2+y^2)} - \left(\frac{c_0^2-2b_0^2}{c_0^2-b_0^2}\right)\frac{(R+N)}{\pi}\frac{x}{(x^2+y^2)} +\\
+ M\theta(x)\delta(y),\\
\sigma_{12}^0(x, y) = -\frac{2G}{\pi}\frac{xy^2}{(x^2+y^2)^2} - \frac{2(R+N)}{\pi}\frac{x^2 y}{(x^2+y^2)^2},\\
u^0(x, y) = \frac{U}{2\rho_0 b_0^2}\left\{\frac{R+N}{\pi}\left[-\frac{c_0^2}{(c_0^2-b_0^2)}\frac{x}{(x^2+y^2)} + \frac{2xy^2}{(x^2+y^2)^2}\right] -\right.\\
\left. - \frac{G}{\pi}\left[\frac{b_0^2}{(c_0^2-b_0^2)}\frac{y}{(x^2+y^2)} + \frac{x^2y-y^3}{(x^2+y^2)^2}\right]\right\},\\
v^0(x, y) = \frac{U}{2\rho_0 b_0^2}\left\{\frac{R+N}{\pi}\left[\frac{b_0^2}{(c_0^2-b_0^2)}\frac{y}{(x^2+y^2)} - \frac{x^2y-y^3}{(x^2+y^2)^2}\right] -\right.\\
\left. - \frac{G}{\pi}\left[\frac{c_0^2}{(c_0^2-b_0^2)}\frac{x}{(x^2+y^2)} + \frac{2xy^2}{(x^2+y^2)^2}\right]\right\}.
\end{gathered} \tag{3.1}$$

Let the force applied to the half-plane be directed vertically downward, i.e., R = 0, G < 0. In this case the elastic part of the solution (3.1) gives the solution of the problem of a concentrated force applied to an elastic half-plane [2]:

$$\begin{gathered}
\sigma_{11}^y(x, y) = -\frac{2G}{\pi}\frac{x^2 y}{(x^2+y^2)^2}, \quad \sigma_{22}^y(x, y) = -\frac{2G}{\pi}\frac{y^3}{(x^2+y^2)^2},\\
\sigma_{33}^y(x, y) = -\left(\frac{c_0^2-2b_0^2}{c_0^2-b_0^2}\right)\frac{G}{\pi}\frac{y}{(x^2+y^2)}, \quad \sigma_{12}^y(x, y) = -\frac{2G}{\pi}\frac{xy^2}{(x^2+y^2)^2}.
\end{gathered} \tag{3.2}$$

We assume that the vertical force applied to the half-plane gives rise to plastic strains described by the Maxwellian model only in a small neighborhood $V = \{(x, y) | x^2 + y^2 \leqslant h^2, y < 0\}$ of the application point. Then, according to (2.3),

$$\varphi_i = -\iint\limits_V \left(\sigma_{ii}^y - \frac{\sigma_{11}^y + \sigma_{22}^y + \sigma_{33}^y}{3}\right) \frac{dxdy}{\tau_s} \quad (i = 1, 2, 3).$$

We use the expressions (3.2) and calculate the values φ_1, φ_2, φ_3:

$$\varphi_1 = \frac{2G}{3\pi\tau_s}\left(\frac{c_0^2 - 2b_0^2}{c_0^2 - b_0^2}\right)h, \quad \varphi_2 = -\frac{2G}{3\pi\tau_s}\left(\frac{c_0^2}{c_0^2 - b_0^2}\right)h, \quad \varphi_3 = \frac{2G}{3\pi\tau_s}\left(\frac{2b_0^2}{c_0^2 - b_0^2}\right)h.$$

We assume that the stress trace arising behind the application point of the force is distributed across a layer of thickness h. Then the magnitude of intensity of the stress layer is determined from the expressions

$$N = \frac{4G}{3\pi\tau_s U}\left(\frac{c_0^2 - 2b_0^2}{c_0^2 - b_0^2}\right)h, \quad M = \frac{2G}{3\pi\tau_s U}\left(\frac{c_0^2}{c_0^2 - b_0^2}\right)h. \tag{3.3}$$

By the condition $G < 0$ for velocities of sound the inequality $c_0^2 > 2b_0^2$ holds; consequently,

$$N < 0, \quad M < 0. \tag{3.4}$$

The dimensions of the plastic zone can be estimated, for example, as is done in [2]. For this we use the Mises yield condition

$$\sqrt{(\sigma_{11}^y - \sigma_{22}^y)^2 + (\sigma_{22}^y - \sigma_{33}^y)^2 + (\sigma_{33}^y - \sigma_{11}^y)^2 + 6(\sigma_{12}^y)^2} = 2\sigma_s,$$

where σ_s is the yield point of the material, and from the elastic solution we find that the plastic zone in this case is a circle $\{x^2 + [y - (d_s/2)]^2 \leqslant d_s^2\}$ whose diameter is given by the expression

$$d_s = -\sqrt{\frac{W}{2}}\frac{G}{\pi\sigma_s} \quad (G < 0), \tag{3.5}$$

where

$$W = 8 - 4\left(\frac{c_0^2 - 2b_0^2}{c_0^2 - b_0^2}\right) + 2\left(\frac{c_0^2 - 2b_0^2}{c_0^2 - b_0^2}\right)^2.$$

Since in reality the load is distributed over an area, although of small but finite width, then in the estimation of the intensity of the stress layer it was assumed that the plastic zone was concentrated in the region V whose depth of penetration was less than the depth of the plastic zone given by the expression (3.5); i.e., $h \leqslant d_s$.

Thus, under the action of a point die on a half-plane moving at a subsonic velocity, along the boundary of the half-plane there arises a stress layer whose intensity is determined by the expressions (3.3), while by virtue of (3.4) the stresses in this layer are compressive.

4. The Stress Trace in the Problem of Impacting Plates under the Conditions of Explosive Welding

Let two plates of equal thickness H collide with one another under the conditions of explosive welding [3]. We approximately assume that the plates in the plane (x, y) connected with the point of contact are depicted by the strips $\{-H < y < 0\}$, $\{0 < y < H\}$. The stress state of such plates is described by Eqs. (2.1) in a coordinate system moving together with the point of contact. The banks of cross sections are directed along the negative part of the real axis and are free from stresses

$$\sigma_{12}(x,0)|_{x<0}=0,\ \sigma_{22}(x,0)|_{x<0}=0. \tag{4.1}$$

The boundary conditions (4.1) and the right sides concentrated in the neighborhood of the point of contact are satisfied if we put $F(z_1)$ and $P(z_2)$ in the expressions (1.4) and (2.2) equal to

$$F(z_1)=a_0\frac{p-iq}{\sqrt{z_1}},\ P(z_2)=-\frac{a_0}{\sqrt{z_2}}\left[p-iq\left(\frac{(1+\omega_2^2)^2}{4\omega_1\omega_2}\right)\right],$$

where

$$p=\frac{\omega_1}{1-\omega_2^2}\left[\left(1-\frac{1+\omega_2^2}{2\omega_1\omega_2}\right)\frac{G}{\pi}+\left(1-\frac{(1+\omega_2^2)^2}{4\omega_1\omega_2}\right)\frac{2\varphi_4}{\pi U}\right];$$

$$q=\frac{1+\omega_2^2}{1-\omega_2^2}\left[\left(1-\frac{1+\omega_2^2}{2\omega_1\omega_2}\right)\frac{(R+N)}{\pi}-\left(1-\frac{(1+\omega_2^2)^2}{4\omega_1\omega_2}\right)\frac{\varphi_2}{\pi U}\frac{b_0^2}{c_0^2}\right];$$

$$a_0=2\Big/\left(1-\frac{(1+\omega_2^2)^2}{4\omega_1\omega_2}\right).$$

The solution with a singularity at $x = y = 0$ presented here describes the character of the stress state near this point. This character must not change if we attain fulfillment of the boundary conditions for $y = \pm H$, which are not taken into account here.

The deformation of the medium takes place without variations of the mass; i.e., the equation

$$U\frac{\partial(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})}{\partial x}-\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0, \tag{4.2}$$

where ε_{11}, ε_{22}, ε_{33} are the components of the strain tensor, is satisfied.

If we assume that in the neighborhood of the point of contact there is an outflow of the mass, then Eq. (4.2) has to be altered to

$$U\frac{\partial(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})}{\partial x}-\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\frac{Q}{c_0^2\rho_0}\delta(x)\,\delta(y), \tag{4.3}$$

where Q/c_0^2 is the power of the outflow of the mass, $Q > 0$.

This outflow can be considered as a modeling cumulative jet which arises in the plastic region at sufficiently high strain rates.

The components of the strain tensor and the stress tensor are connected with one another by Hooke's law. Then

$$\sigma_{11}+\sigma_{22}+\sigma_{33}=3\rho_0\left(c_0^2-\frac{4}{3}b_0^2\right)(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})$$

and Eq. (4.3) assumes the form

$$U\frac{\partial(\sigma_{11}+\sigma_{22}+\sigma_{33})}{\partial x}-3\left(c_0^2-\frac{4}{3}b_0^2\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=3\left(c_0^2-\frac{4}{3}b_0^2\right)\frac{Q}{c_0^2}\delta(x)\,\delta(y).$$

If there is an outflow of the mass, then the conditions (2.4) must be changed to

$$\varphi_1 + \varphi_2 + \varphi_3 = 3\left(c_0^2 - \frac{4}{3}b_0^2\right)\frac{Q}{c_0^2}. \tag{4.4}$$

Let the relaxation take place so that the powers of the relaxation sources of the stresses satisfy the relations

$$\varphi_1 = \varphi_3, \; \varphi_2 > 0. \tag{4.5}$$

If φ_i ($i = 1, 2, 3$) satisfy the conditions (4.4), (4.5), then

$$\varphi_1 = 3\left(c_0^2 - \frac{4}{3}b_0^2\right)\frac{Q}{2c_0^2} - \frac{\varphi_2}{2}. \tag{4.6}$$

A stress trace remains behind the point of contact along the positive part of the real axis.

The stresses in this layer are determined by the quantities N and M. In the case (4.5) N = M. Using the expression (4.6) we obtain

$$N = 3\left(c_0^2 - \frac{4}{3}b_0^2\right)\frac{(Q - \varphi_2)}{2c_0^2}. \tag{4.7}$$

We shall investigate the expression thus obtained. If the power of the mass outflow is zero, then

$$N = -\frac{3}{2c_0^2}\left(c_0^2 - \frac{4}{3}b_0^2\right)\varphi_2,$$

and since in view of (4.5) $\varphi_2 > 0$, then $N < 0$. This means that if in the problem of impact of two plates there is no outflow of the mass in the neighborhood of the point of contact, then on the line of the joint of the plates there is formed a stress trace subjected to compressive stresses.

Let in the neighborhood of the point of contact there be an outflow of the mass, with its power Q being equal to the power φ_2 of the source of relaxation of the normal stress. Then $N = 0$ and, consequently, after the point of contact no stress trace is formed; i.e., the presence of the outflow of the mass, which in power is equal to the power φ_2 of relaxation of the normal stress, removes the stress layer in the neighborhood of the point of contact.

We note that in [4] it is experimentally established that the phenomena of wave formation and cumulative jet formation, as a rule, mutually exclude one another. In the zone of parameters corresponding to stable formation of a cumulative jet no welding waves are usually observed, and conversely, under the conditions of wave formation clearly discernable signs of a jet are absent.

Apparently, this is connected with the circumstance that according to (4.7) the outflow of the mass removes the stresses in the trace after the point of contact. In this case it is natural to expect that the presence of the stress trace is the cause of the wave formation.

The problem of impacting plates in an elastic formulation was investigated in [5], where the plastic strains in the neighborhood of the point of contact were not taken into account and, therefore, the formation of a stress trace behind the point of contact was not noted.

5. Numerical Estimate of the Quantities U and γ Necessary for Stability Loss of the Stress Layer

Let the stress trace arising behind the point of contact be distributed over the layer of thickness 2h. Then the intensity of this stress layer can approximately be estimated as follows. We assume that the action of one plate on the other is the action of a point die on a moving half-plane whose velocity coincides with the velocity of the point of contact. The force with which the die acts on the half-plane is determined from the law of conservation of the vertical impulse. It is determined from the expression

$$G = \rho_0 U^2 H \sin(\gamma/2), \tag{5.1}$$

where H is the width of the plate; γ is the angle of impact. In the following we shall assume that the force is concentrated in the neighborhood of the point of contact.

With such an assumption the estimate of the intensity of the stress layer in the problem of plate impact under the conditions of explosive welding reduces to the estimate, carried out above, of the intensity of the stress layer that arises as a result of the action of a point die on a moving half-plane.

Following the idea proposed in [6], we consider the following model problem. Let the stress layer arising on the line of contact when impact welding two plates be a two-layered rod; let each of the rods be subjected to compression with an intensity N. On the line separating the rods, at a sufficient distance from the point of contact, the stresses σ_{12} and σ_{22} are zero in view of symmetry of the problem. We shall assume that the wave formation during explosive welding is the loss of stability of the rod compressed by a longitudinal load 2N and attached to the elastic half-plane. In essence this hypothesis consists of the fact that the wave formation during explosive welding has a cause which is completely analogous to the cause of star-shaped deformations of tubes close to detonating charges of explosives described in [7].

To confirm this hypothesis we have to determine whether U and γ exist, for which the stress layer is compressed by a load equal to or exceeding the critical load, and how close these U and γ are to the experimental values.

The differential equation for transverse flexure of a rod of height h, compressed by a load $T_0 h$ and fixed to an elastic foundation has the form

$$\frac{E_0 h^3}{12(1-\sigma_0^2)} \frac{d^4\xi}{\partial x^4} + T_0 h \frac{d^2\xi}{\partial x^2} + R = 0, \tag{5.2}$$

where E_0 is Young's modulus of the rod being compressed; σ_0 is the Poisson ratio; R is the reaction of the elastic foundation.

Let after the loss of stability the rod have the flexure

$$\xi(x) = A e^{i\alpha x} \ (A - \text{const}).$$

For such flexure the value of the reaction of the elastic half-plane is given by the expression

$$R = -\frac{E\alpha}{2(1-\sigma^2)} \xi(x), \tag{5.3}$$

where E is Young's modulus of the elastic foundation; σ is the Poisson ratio.

We substitute the value of R given by the expression (5.3) into Eq. (5.2); after a transformation we obtain

$$\left(\frac{h}{\lambda}\right)^3 + p\left(\frac{h}{\lambda}\right) + q = 0, \tag{5.4}$$

where $\lambda = 2\pi/\alpha$ is the length of the wave into which the rod is bent;

$$p = -\frac{3(1-\sigma_0^2)}{\pi^2 E} T_0; \ q = -\frac{3}{4\pi^3} \frac{E(1-\sigma_0^2)}{E_0(1-\sigma^2)}.$$

TABLE 1

D_0	Cu			Fe		
	P_0	λ/h	γ^0_*	P_0	λ/h	γ^0_*
10	0,161	2,0	13	0,183	2,0	17
30	0,112	1,4	9	0,127	1,4	12
60	0,09	1,1	8	0,1	1,1	10
90	0,08	1	7	0,088	1	9

TABLE 2

Material	ρ_0 g/cm³	c_0, km/sec	b_0, k/sec
Cu	8,900	4,651	2,1409
Fe	7,840	5,694	2,8659

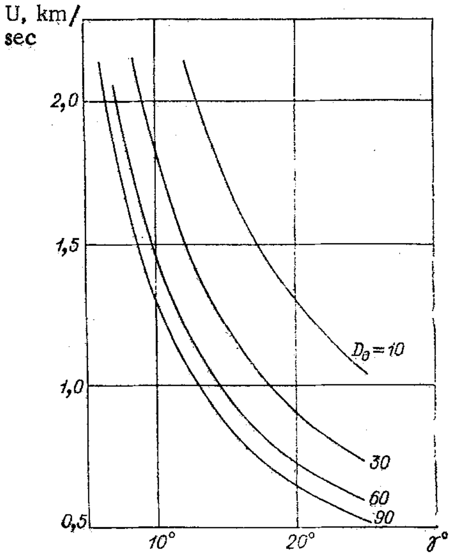

Fig. 1

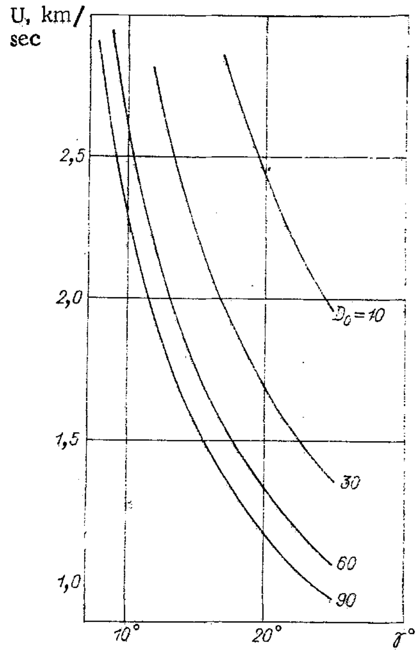

Fig. 2

We introduce the notation

$$D_0 = \frac{E\left(1-\sigma_0^2\right)}{E_0\,(1-\sigma^2)},\; P_0 = \frac{T_0}{\rho_0 c_0^2}.$$

In this notation the coefficients p and q of the cubic equation (5.4) are written in the form

$$p = -\frac{3P_0\rho_0 c_0^2}{\pi^2}\left(\frac{1-\sigma^2}{E}\right)D_0,\; q = -\frac{3}{4\pi^3}D_0.$$

Thus, the problem of stability loss of the rod fixed to an elastic foundation and being compressed by a longitudinal load T_0h has been reduced to finding, for a fixed parameter D_0, of the value of the parameter P_0 for which Eq. (5.4) has a double real root.

Solving the prcblem thus obtained, we find the expression for P_0 dependent on D_0:

$$P_0 = \sqrt[3]{\frac{9}{D_0}}\,\frac{E}{4\rho_0 c_0^2\,(1-\sigma^2)}. \tag{5.5}$$

Since on the line of contact of the plates the material deforms plastically, Young's modulus of the rod in compression assumes a certain effective value which in magnitude does not exceed the modulus of elasticity of the material; i.e., $E_0 < E$ and, consequently, $D_0 > 1$.

The critical values of P_0 determined by (5.5) and the corresponding wavelengths into which the rod is bent are presented for copper and iron in Table 1. The values of $\rho_0 c_0$, b_0 are given in Table 2.

TABLE 3

D_0 \ τ_s/τ	Cu					Fe				
	1	2	3	4	5	1	2	3	4	5
10	53	33	24	19	16	52	32	24	19	15
30	65	43	32	26	21	64	42	31	25	21
60	72	49	38	31	26	72	49	37	30	26
90	76	53	41	33	28	76	53	41	33	28

TABLE 4

γ° \ U, km/sec	Fe						Cu					
	1,56	1,66	1,77	1,9	2,04	2,21	0,85	0,9	0,96	1,03	1.11	1,2
17	20	22	26	29	34	40	19	21	25	28	32	37
16		24	27	31	36	42		22	25	30	34	39
15			29	33	39	45			27	32	36	42
14				36	41	48				34	38	45
13					44	52					41	49
12						57						53

In the case of the problem of impact of two plates under the conditions of explosive welding, the quantity P_0 (in view of the assumptions made above) is specified by the expression

$$P_0 = \frac{4G}{3\pi\tau_s\rho_0 c_0^2 U}\left(\frac{c_0^2 - 2b_0^2}{c_0^2 - b_0^2}\right), \tag{5.6}$$

where G is given by the relation (5.1).

The entire stress layer is compressed by the load 2N and has the height 2h, but since at a sufficient distance from the point of contact the boundary of the rods is free from stresses, then, speaking of the critical load and the wave length along which the layer (the rod) is bent after the loss of stability, it is sufficient to consider the stress layer as a single rod of height h fixed to the elastic half-plane and being compressed by the longitudinal load N.

In the (U, γ) plane we construct a curve that separates the region of values of U and γ for which wave formation is possible. Let P_0 be the critical value for which the stress layer loses its stability. Then the expression (5.6) gives the following connection between U and γ:

$$U = \frac{3\pi\tau_s P_0 c_0^2}{4H}\left(\frac{c_0^2 - b_0^2}{c_0^2 - 2b_0^2}\right)\frac{1}{\sin(\gamma/2)}. \tag{5.7}$$

Points of the (U, γ) plane lying above the curve (5.7) correspond to the values of the velocity of points and the angle of impact for which wave formation is possible. The following constraints are imposed on U and γ. Since a subsonic impact of the plates is considered ($U^2 < b_0^2$), then, for the stress layer to be compressed by a load that is equal or exceeds the critical load, it is necessary for the angle of impact to be greater than a certain angle γ_*. On the other hand, at high velocities of the point of contact and large angles of impact the metal on the surface of the colliding plates can be regarded as liquid. In [8, 9] in the framework of the theory of a viscous liquid an inequality involving U and γ is obtained for which jet formation is possible:

$$\frac{HU}{\nu}\frac{\sin^2(\gamma/2)}{[1 - \sin(\gamma/2)]} > 2, \tag{5.8}$$

where ν is the kinematic coefficient of viscosity, $\nu = b_0^2\tau$, and τ is the relaxation time of shear stresses.

In the case where the metal on the plate surface behaves as a liquid, $\tau < \tau_s$.

Having used (5.7), (5.8), we obtain

$$\sin(\gamma/2) = \left[1 + \frac{3\pi P_0 c_0^2 \tau_s}{8 b_0^2 \tau}\left(\frac{c_0^2 - b_0^2}{c_0^2 - 2b_0^2}\right)\right]^{-1}. \quad (5.9)$$

From (5.9) we determine the critical angle γ_{**} such that for $\gamma > \gamma_{**}$ jet formation is possible ("outflow of the mass"), and this means that on the line of contact the stress layer is removed. Since we assume that the stress layer is the cause of wave formation, then it is natural that in the case $\gamma > \gamma_{**}$ there will be no wave formation.

In Figs. 1 and 2 we have presented the graphs of the curve (5.7) for copper and iron, respectively, for different D_0 ($\tau_s = 0.1$ µsec). The angle γ_* and the ratio of the wavelength into which the layer is bent to the thickness of the layer, $k = \lambda/h$, are given in Table 1. The values of the angle γ_{**} for different values of the parameter D_0 and τ_s/τ are given in Table 3. The height of the stress layer is specified by the expression

$$h = \beta d_s,$$

where β is a numerical parameter not exceeding unity; d_s is given by the expression (3.5).

Then for the wavelength we obtain the relation

$$\lambda = \beta k d_s. \quad (5.10)$$

We shall assume that β varies within the limits from 0.1 to 0.5, k varies from 1 to 2 and, consequently, the coefficient βk of d_s in the expression (5.10) varies from 0.1 to 1.

In [3] for the wavelength obtained in the case of explosive welding the experimental expression

$$\lambda = 26H \sin^2(\gamma/2) \quad (5.11)$$

is given.

The expression (5.10) can be written in the form (5.11)

$$\lambda = LH \sin^2(\gamma/2),$$

where

$$L = \beta k \frac{\rho_0 U^2}{\pi\sigma_s}\sqrt{\frac{W}{2}}\frac{1}{\sin(\gamma/2)}.$$

The quantity L depends on the density of the material, the yield point, the velocity of the point of contact, and the angle of impact. We put $\beta k = 0.2$ and find the values of L in the case of different U and γ for copper [$\sigma_s = 6.5$ kbar (1 kbar $= 10^8$ N/m^2)] and iron ($\sigma_s = 20$ kbar). These values are presented in Table 4.

LITERATURE CITED

1. L. A. Galin, "A mixed problem of the theory of elasticity with friction forces for a half-plane," Dokl. Akad. Nauk SSSR, 39, No. 3 (1943).
2. V. S. Smirnov, The Theory of Working Metals by Pressure [in Russian], Metallurgiya, Moscow (1973).
3. M. A. Lavrent'ev and B. V. Shabat, Problems of Hydrodynamics and Their Mathematical Models [in Russian], Nauka, Moscow (1973).

4. V. A. Simonov, "On flows arising when a shock wave falls onto a wedge-shaped plane," Fiz. Goreniya Vzryva, 7, No. 2 (1971).
5. V. V. Efremov, "An investigation of skew impacts of metal plates in an elastic formulation," Zh. Prikl. Mekh. Tekh. Fiz., No. 1 (1975).
6. L. B. Érlikh, V. A. Kosobludskii, and L. I. Vershinin, Wave Formation on Smoothed Surfaces [in Russian], Nauka, Moscow (1973).
7. M. A. Lavrent'ev and A. Yu. Ishlinskii, "Dynamic modes of loss of stability of elastic systems," Dokl. Akad. Nauk SSSR, 65, No. 6 (1949).
8. S. K. Godunov and A. A. Deribas, "On the problem of jet formation in metal impact," Dokl. Akad. Nauk SSSR, 202, No. 5 (1972).
9. S. K. Godunov, A. A. Deribas, and V. I. Mali, "On the influence of viscosity of the material on the process of jet formation during an impact of metal plates," Fiz. Goreniya Vzryva, No. 1 (1975).

PROPAGATION OF A TWO-DIMENSIONAL PLASTIC WAVE IN A NONLINEARLY COMPRESSED HALF-PLANE

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UDC 539.374:534.1

We shall consider a two-dimensional stationary problem of the propagation of a shock wave in a nonelastic ideal medium filling a half-space, when a moving load acts on its boundary. The solution of the problem is constructed by the method of characteristics for the case where the velocity of motion of the load exceeds the velocity of propagation of the shock wave in the medium whose compressibility is nonlinear and irreversible (Fig. 1), while in [1] the case of a linear relation between p and ε is investigated analytically. At the same time the surface of the medium where the pressure is applied is assumed, just as in [2, 3], only little deformed, and therefore it is assumed that the pressure is applied to a horizontal nondeformed surface (Fig. 2).

The scheme proposed provides us with a possibility of carrying out the calculation of the parameters of the medium (in particular, of the ground) that is being modeled by a generalized plastic gas [2] or an ideal liquid [4], and also in the case of wave propagation in reservoirs with a screen [3], and so forth.

The results of the numerical calculation are represented in the form of curves of the variation of the pressure and the velocity of the medium in the region of perturbation along the wave front.

Let a monotonically decreasing normal pressure move along the surface of a half-space at a velocity D (see Fig. 2). Then in the half-space a shock wave with a curvilinear front Σ will propagate at a velocity a, the value of which is not known in advance and is determined in the solution process of the problem.

We shall assume that the medium on the front Σ is instantaneously loaded, while behind the front (in the perturbed region) there occurs unloading which is assumed to be linear. In this case on the front Σ from the condition of conservation of the mass and the impulse we obtain

$$\rho_0 a = \rho^* (a - v_n^*),\ \rho_0 a v_n^* = p^*,\ v_\tau^* = 0, \tag{1}$$

while the equation of state of the medium is represented in the form of a polynomial

$$p^* = \alpha_1 \varepsilon^* + \alpha_2 \varepsilon^{*2}. \tag{2}$$

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